Refraction and gravity

October 2020

1 Motivation

The goal is to obtain an expression for the refraction angle of (monochromatic) light that bends not only through gravity governed by general relativity, but also by a medium that surrounds the body of interest. In other words, we elaborate on classical light diffraction under the laws of general relativity, i.e. in curved space-time. We expect the refraction angle to depend on material dependent quantities, which can then be determined by measurements, yielding information about stellar or planetary atmospheres and their composition.

2 Derivation

We choose the weak field approximation, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $g_{\mu\nu}$ is the metric tensor, $\eta_{\mu\nu}$ the one of flat Minkwoski space and $h_{\mu\nu}$ a small perturbation, which means that

$$h_{\mu\nu} << \eta_{\mu\nu}. \tag{2.2}$$

We use the Schwarzschild metric

$$ds^2 = \left(1 - \frac{\beta}{r}\right) c^2 dt^2 - \left(1 + \frac{\beta}{r}\right) dl^2, \tag{2.3}$$

where c denotes the speed of light in vacuum, $\beta = \frac{2GM}{c^2}$ is the Schwarzschild radius, $dl = \sqrt{dx^2 + dy^2 + dz^2}$, and $r = \sqrt{x^2 + y^2 + z^2}$.
We want to look at a monochromatic light beam, thus we get null geodesics

$$ds^2 = 0. \tag{2.4}$$

Combining this with equation 2.3 and rearranging yields

$$\frac{dl}{dt} = c\sqrt{\frac{1 - \frac{\beta}{r}}{1 + \frac{\beta}{r}}}. \tag{2.5}$$

So far we have only repeated the calculations of `https://arxiv.org/pdf/1709.04127.pdf` section 2.
If we now assume that the light traveling around the center of mass is not only propagating through vacuum but rather through matter with a refraction index n, it stands to reason to modify the last equation in the following way

$$v = \frac{dl}{dt} = \frac{c}{n}\sqrt{\frac{1-\frac{\beta}{r}}{1+\frac{\beta}{r}}}. \tag{2.6}$$

This specific modification can also be justified by the classical limit in which the square root becomes simply 1 and we obtain the well known relation $v = \frac{c}{n}$.
Without limiting the generality we can choose some coordinates in a useful way such that the calculation for the deflection angle will be more easy. Thus we look only at the $z = 0$ plane and name the impact parameter $y = b$ at $x = 0$. The following figure illustrates the definitions.

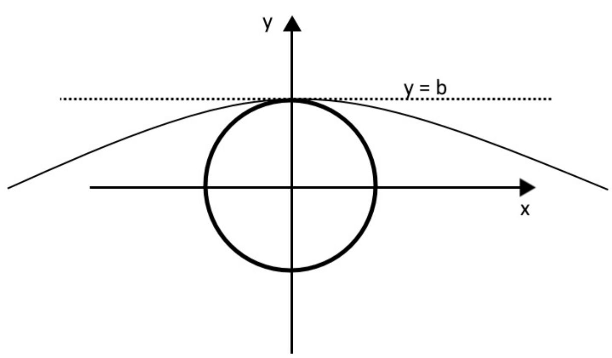

Figure 2.1: Schematic diagram

All in all we get by using the formula derived by *Einstein* for the deflection angle using Fermat's principle

$$\Theta = \frac{1}{c}\int_{-\infty}^{+\infty}\left(\frac{dv}{dy}\right)\bigg|_{y=b} dx \tag{2.7}$$

However, an exact determination of the value of 2.7 is impossible. But for celestial bodies the parameter $\frac{\beta}{r}$ becomes very small and so we can approximate

$$\frac{1}{1+\frac{\beta}{r}} \approx 1 - \frac{\beta}{r}. \tag{2.8}$$

For now the refraction index did not play any role. We shall treat two cases here, where one is more relevant than the other.

2.1 Barometric density

In order to introduce the refraction index n, we first need the link between the matter density and the refraction index since the matter density of atmospheres is mostly well known. For that we use the Clausius-Mossotti equation that reads

$$P = \frac{\epsilon_r - 1}{\epsilon_r + 2}\frac{M_m}{\rho}, \tag{2.9}$$

where P denotes the polarizability of the material, $\epsilon_r = |n|^2$ the dielectric function, M_m the molar mass and ρ the matter density. Rearranging previous relation yields

$$n = \pm i\sqrt{\frac{M_m + 2P\rho}{P\rho - M_m}}. \tag{2.10}$$

Thus, we have expressed the refraction index by the matter density and only by material dependent constants. For better handiness we use the Taylor expansion for the matter density which is given by

$$\rho(r) = \rho_0 e^{-\frac{r}{H_p}}, \tag{2.11}$$

where ρ_0 is the density at the surface of the object and H_p is the scalar height; both quantities are constants. Furthermore, r denotes again the coordinate distance from the center of the body. Now, by making the justified assumption that $e^{-\frac{r}{H_p}} << \frac{3P\rho_0}{2M_m}$, equation (2.10) becomes

$$\frac{1}{n} = 1 - \alpha e^{-\frac{r}{H_p}} + o(\rho^2), \tag{2.12}$$

where we defined the matter dependent constant $\alpha = \frac{3P\rho_0}{2M_m}$. We already inverted the refraction index as this is the quantity that is relevant for the velocity and therefore ultimately for the refraction angle.

Now we are able to calculate the refraction angle given by the previous relation (2.3):

$$\Theta = \frac{1}{c}\int_{-\infty}^{+\infty}\left(\frac{dv}{dy}\right)\Big|_{y=b} dx \tag{2.13}$$

$$\approx \frac{d}{dy}\int_{-\infty}^{+\infty} dx\left(1 - \alpha e^{-\frac{r}{H_p}}\right)\left(1 - \frac{\beta}{r}\right)\Big|_{y=b} \tag{2.14}$$

$$= \frac{d}{dy}\int_{-\infty}^{+\infty} dx\left(1 - \frac{\beta}{r} - \alpha e^{-\frac{r}{H_p}} + \beta\frac{\alpha e^{-\frac{r}{H_p}}}{r}\right)\Big|_{y=b} \tag{2.15}$$

(2.16)

The first two terms are the unperturbed refraction angle, i.e. without any medium and yields the known result $\Theta_1 = \frac{2\beta}{b}$. The other terms read

$$\Delta\Omega = \alpha \frac{d}{dy} \int_{-\infty}^{+\infty} dx \left(e^{-\frac{r}{H_p}} + \beta \frac{\alpha e^{-\frac{r}{H_p}}}{r} \right) \Big|_{y=b}. \tag{2.17}$$

Both terms are even functions of x, thus

$$\Delta\Omega = 2\alpha \frac{d}{dy} \int_{0}^{+\infty} dx \left(e^{-\frac{r}{H_p}} + \beta \frac{\alpha e^{-\frac{r}{H_p}}}{r} \right) \tag{2.18}$$

Also, both integrals are analytically solvable by using the the substitution $x = y\cosh(t)$. We therefore have the relation

$$dx\, e^{-\frac{\sqrt{x^2+y^2}}{H_p}} = dt\, y \cosh(t) e^{-y\frac{\cosh(t)}{H_p}}. \tag{2.19}$$

Hence, by using the standard integral

$$\int_{0}^{+\infty} dt e^{-y\frac{\cosh(t)}{H_p}} \cosh(\nu t) = J_\nu(z), \tag{2.20}$$

where $J_\nu(z)$ denotes the Bessel function, the first term becomes

$$\int_{0}^{+\infty} dx\, e^{-\frac{r}{H_p}} = y J_1\left(\frac{y}{H_p}\right). \tag{2.21}$$

In a similar fashion we obtain the second term of (2.18) as

$$\int_{0}^{+\infty} dx \frac{e^{-\frac{r}{H_p}}}{r} = J_0\left(\frac{y}{H_p}\right). \tag{2.22}$$

By collecting all terms we end up with the final result

$$\Delta\Omega = 2\alpha \frac{d}{dy} \left(y J_1\left(\frac{y}{H_p}\right) + \beta J_0\left(\frac{y}{H_p}\right) \right) \Big|_{y=b}. \tag{2.23}$$

In the limit of sufficiently small $\frac{b}{H_p}$, this can be expanded and the derivative can be applied, i.e.

$$\Theta = \alpha \frac{b}{H_p} \left(2 - \frac{\beta}{H_p} \right) + o\left(\frac{b}{H_p}\right)^2, \tag{2.24}$$

yielding the total refraction angle

$$\boxed{\Theta = \Theta_1 + \Delta\Theta = \tfrac{2\beta}{b} + \alpha \tfrac{b}{H_p}\left(2 - \tfrac{\beta}{H_p}\right).} \tag{2.25}$$

Just like we expected, the correction depends on the material constant α, containing the polarizability, molecular mass and also on the surface matter density.

2.2 Polytropes

In general, one can derive the correction angle for a polytropic gas, which is often used in astronomical models. It is defined by the relation

$$P = K\rho^{\frac{\gamma+1}{\gamma}}, \tag{2.26}$$

where P stands for the pressure and K for the polytropic constant.

To illsutrate this, let us consider a refraction index of a polytropic gas with $\gamma = 1$. The relevant equation that has to be considered is the stellar structure equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{Kr^2}. \tag{2.27}$$

Solving this simple differential equation for the case $\gamma = 1$, we obtain that $\rho \propto \frac{1}{r}$. Further, we simplify this consideration by stating that $n \propto \rho(r)$. Moreover, we assume it to be slightly larger than the one of vacuum $n_{vac} = 1$ and we want it to depend on the distance r to the center

$$n(r) = 1 + \frac{n_0}{r}, \tag{2.28}$$

where n_0 is a medium dependent constant. We see that for large distances from the body $r >> 1$ the refraction index $n(r)$ becomes approximately the one of vacuum as it should be.
In the same manner as $\frac{\beta}{r}$ we approximate the refraction index by arguing that $\frac{n_0}{r}$ is small. This is reasonable because for most bodies of interest, for instance stars, the corona (not the virus) density usually falls off very fast with distance. Therefore

$$\frac{1}{1+\frac{n_0}{r}} \approx 1 - \frac{n_0}{r}. \tag{2.29}$$

Therefore by combining equation 2.7 with the last two approximations we obtain by using the product rule

$$\Theta = \frac{1}{c}\int_{-\infty}^{+\infty}\left[\frac{d}{dy}\left(c\left(1-\frac{n_0}{r}\right)\left(1-\frac{\beta}{r}\right)\right)\right]\Big|_{y=b} dx \tag{2.30}$$

$$= \int_{-\infty}^{+\infty}\left[\left(\frac{n_0 b}{(x^2+b^2)^{\frac{3}{2}}}\right)\left(1-\frac{\beta}{(x^2+b^2)^{\frac{1}{2}}}\right)+\left(1-\frac{n_0}{(x^2+b^2)^{\frac{1}{2}}}\right)\left(\frac{\beta b}{(x^2+b^2)^{\frac{3}{2}}}\right)\right] dx \tag{2.31}$$

$$= \int_{-\infty}^{+\infty}\left[\frac{n_0 b}{(x^2+b^2)^{\frac{3}{2}}} - \frac{\beta n_0 b}{(x^2+b^2)^2} + \frac{\beta b}{(x^2+b^2)^{\frac{3}{2}}} - \frac{\beta n_0 b}{(x^2+b^2)^2}\right] dx \tag{2.32}$$

$$= \int_{-\infty}^{+\infty}\left[\frac{b(\beta+n_0)}{(x^2+b^2)^{\frac{3}{2}}} - \frac{2\beta n_0 b}{(x^2+b^2)^2}\right] dx. \tag{2.33}$$

Now by substituting we obtain

$$\xi = \frac{x}{b} \tag{2.34}$$

$$\Theta = \int_{-\infty}^{+\infty} \left[\frac{\beta + n_0}{b(\xi^2+1)^{\frac{3}{2}}} - \frac{2\beta n_0}{b^2(\xi^2+1)^2} \right] \Big|_{y=b} d\xi \tag{2.35}$$

$$= \frac{(2b - \pi\beta)n_0}{b^2} + \frac{2\beta b}{b^2}, \tag{2.36}$$

which results in

$$\boxed{\Theta = \frac{2\beta}{b} + \frac{(2b-\pi\beta)n_0}{b^2}.} \tag{2.37}$$

We can see the deflection angle without a medium, i.e. $n_0 = 0$ yields the well known relation

$$\Theta_0 = \frac{2\beta}{b}. \tag{2.38}$$

We plug in some dummie values

$$M = M_\odot \tag{2.39}$$

$$b = R_\odot \tag{2.40}$$

$$n_0 = 1, \tag{2.41}$$

where it is striking that the extra term $\frac{n_0}{r}$ is indeed very small, thus fulfilling our assumption. We obtain

$$\Theta \approx 8.8487 \cdot 10^{-6}, \tag{2.42}$$

while for $n_0 = 0$ we get the famous angle

$$\Theta \approx 8.8484 \cdot 10^{-6}. \tag{2.43}$$

Hence only an optically thin medium with $n(r) = 1 + \frac{n_0}{r}$ with $\frac{n_0}{r} << 1$ around a massive body already yields corrections. If we plug in optically thicker media we obtain even more significant corrections.

3 Discussion

We now look at the recent data for deflection angles and *Einstein*'s prediction in the figure below.

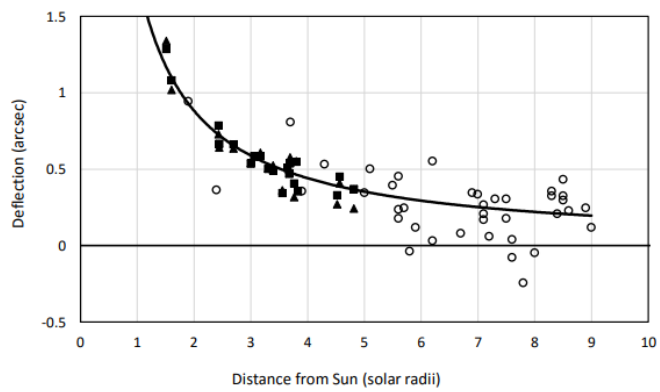

Figure 3.1: The deflection measurements for all 20 stars are plotted as a function of radial distance from the Sun. The solid curve follows the theoretical value of 1.751 arcsec. The triangles mark the Astrometrica results and the squares mark the MaxIm DL results. For comparison, the results from the 1973 experiment are shown as open circles. | Source: Donald G. Bruns 7387 Celata Lane, San Diego, CA 92129

Indeed close to one solar radius the measured deflection angle is above the one predicted by Einstein in the order of 0.1arcsec, which could potentially be modelled by the approach above by choosing a fitting $n(r)$ or a fitting n_0 in the case of polytropes. That the order of derivation fits with small adjustments to n_0 can be seen by expressing the results 2.42 and 2.43 in terms of arcseconds.
Moreover, it is possible to choose a completely new approach for $n(r)$ to model other dependencies

4 Application

The scope of application for this method is extremely broad. One can use it for instance to model any kind of atmosphere, either the one of stars, planets or the ones of any celestial body which is covered with a medium. This implies also in particular the model of protoplanetary systems, where gaseous clouds result in areas with different optical thicknesses, leading to different refraction indices and therefore different refraction angles.